

Lift and drag evaluation in translating and rotating non-inertial systems

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Abstract

In this paper relationships have been derived for lift and drag coefficients for cylindrical bodies for two cases. The relative motion between the body and the fluid is assumed to be two-dimensional and to take place in a plane perpendicular to the axis of the body. Three-dimensional effects are ignored, thus limiting the validity of the formulae to low Reynolds number flows. The fluid is assumed to be an incompressible constant-property Newtonian fluid. In the first case, an inertial system is fixed to a stationary cylindrical body. The motion of the fluid in which the body is placed is an arbitrary function of time not identically zero, e.g. the fluid can have linear and angular acceleration, such as translation, oscillation or rotation. The velocity of the fluid at a single instant is either uniform in space or, in the case of rotation, a linear function of distance from the origin of the system. In the second case, a noninertial system is fixed to an accelerating cylindrical body. The relative flow between fluid and body is kinematically the same as in the first case, but the forces acting upon the bodies differ in the two systems. This is due to the inertial forces that occur in a noninertial system. General formulae are derived for a cylindrical body of arbitrary cross-section and give the relationships between the two systems for each set of coefficients, i.e. the relationship between the lift coefficients for each case, and the same for the drag coefficient. As an example, the relationships are applied to two common cases, a circular and a rectangular cross-section cylinder.

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1. Introduction

Recently there has been renewed interest in investigating flow around cylindrical bodies (circular, rectangular, and of various other shapes). Such investigations have been approached experimentally, e.g. Norberg (2003), Roshko (1993), Williamson (1996), as well as numerically, e.g. Karniadakis and Triantafyllou (1989), Posdziech and Grundmann (2001). In addition to these studies of uniform flow around stationary bodies, there have been a growing number of studies dealing with uniform flow around bodies in motion, such as oscillating or rotating bodies, e.g. Blackburn and Henderson (1999); Chew et al. (1995), bodies undergoing rotary oscillation, e.g. Mahfouz and Badr (2000), Cheng et al. (2001), or bodies in orbital motion, e.g. Williamson et al. (1998); Baranyi (2003). Furthermore, researchers have also been investigating stationary bodies placed in oscillatory flow [see e.g. Bearman et al. (1985); Sarpkaya (1986)]; while Li et al. (2002) investigated fluid/structure interaction in a moving frame of reference. A further variation includes the

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Nomenclature

$\mathbf{a}_0 = a_{0x}\mathbf{i} + a_{0y}\mathbf{j}$ cylinder acceleration, nondimensionalized by U^2/l
 C bounding curve of cross-section of cylindrical body
 C_D drag coefficient
 C_L lift coefficient
 D length of rectangle in x direction, nondimensionalized by l
 \mathbf{e}_s unit vector tangential to curve C , see Fig. 1
 \mathbf{e}_n unit vector normal to curve C , see Fig. 1
 H height of rectangle, nondimensionalized by l
 $\mathbf{i}, \mathbf{j}, \mathbf{k}$ unit vectors in x, y, z and/or ξ, η, ψ directions, respectively
 l length scale
 n coordinate normal to curve C , nondimensionalized by l
 p pressure, nondimensionalized by ρU^2
 $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ position vector in the inertial system, nondimensionalized by l
 $\mathbf{r}_r = \xi\mathbf{i} + \eta\mathbf{j} + \psi\mathbf{k}$ position vector in the noninertial system, nondimensionalized by l
 $r_r = \sqrt{\xi^2 + \eta^2}$ distance measured from the origins of systems in ξ, η plane, nondimensionalized by l
 Re Reynolds number, Ul/ν
 s arc-length along contour C , nondimensionalized by l
 t time, nondimensionalized by l/U
 U velocity scale
 $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$ velocity vector, nondimensionalized by U
 $\mathbf{w} = w_\xi\mathbf{i} + w_\eta\mathbf{j}$ relative velocity vector, nondimensionalized by U
 ζ vorticity, nondimensionalized by U/l
 Θ dilation, nondimensionalized by U/l
 ν kinematic viscosity
 ρ fluid density
 τ shear stress, nondimensionalized by ρU^2
 φ polar angle, see Fig. 2
 χ angle included between unit vector \mathbf{e}_s and x -axis, see Fig. 1
 $\boldsymbol{\omega} = \omega\mathbf{k}$ angular velocity vector, nondimensionalized by U/l
 ∇ del operator, nondimensionalized by $1/l$
 ∇^2 Laplace operator, nondimensionalized by $1/l^2$

Superscripts

$*$ in inertial system
 \sim dimensional quantity

Subscripts

A in inertial system
 B, r in noninertial or relative system
 u upper
 x, y, ξ, η, s, n components in x, y, ξ, η, s, n, s and n directions, respectively

investigation of flow and forces in an oscillating or rotating body in an otherwise quiescent fluid [e.g. Guilmineau and Queutey (2002)].

Amidst these variations, Bearman et al. (1995) considered particularly the lift coefficient for two ambient flow conditions. In the first case a stationary cylinder is exposed to the superposition of a uniform flow and a transversely oscillating flow, and in the second a transversely mechanically oscillated cylinder is placed in a uniform stream. Although the two flows are kinematically identical, they are different dynamically. Bearman et al. chose to calculate the

first case, to avoid the complication of adapting the mesh to take into account the cylinder motion, and from this they derive the lift coefficient for the case when the cylinder is oscillating. An extra force is added to the lift computed for the case of a stationary cylinder in a transversely oscillating flow. This addition to the lift force is not generated when there is just in-line oscillation of the flow.

Now the author, like Bearman et al. (1995), would like to consider two cases that are kinematically identical with each other. However, a much more general motion of the cylinder is assumed here: not only transverse but also in-line oscillation and rotation or angular acceleration as well. The cross-section of the cylinder is also not limited to a circle but can be arbitrary in this study. The undisturbed flow far from the cylinder can be an arbitrary function of time not identically zero. The velocity of the fluid at a single instant is either uniform in space or, in the case of rotation, a linear function of distance from the origin of the system, which is the point around which the cylindrical body is rotated. The fluid is assumed to be an incompressible, constant-property Newtonian fluid. The relative motion between the body and the fluid is assumed to be two-dimensional and to take place in a plane perpendicular to the axis of the body. Three-dimensional effects are ignored, thus limiting the validity of the formulae to low Reynolds number flows.

The purpose of this paper is to derive relationships for lift and drag coefficients defined in an inertial system and in a noninertial system fixed to a moving cylindrical body. Equations will be derived for a general case from which special cases can easily be obtained.

2. Derivation of lift and drag coefficients in an inertial system

Before starting to derive the basic equations let us define two vectors. Unit vectors \mathbf{e}_s and \mathbf{e}_n are tangential and normal to contour \tilde{C} of a cylindrical body of arbitrary cross-section and of unit span (see Fig. 1), respectively

$$\mathbf{e}_s = \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{s}} = \frac{\partial \tilde{x}}{\partial \tilde{s}} \mathbf{i} + \frac{\partial \tilde{y}}{\partial \tilde{s}} \mathbf{j} = \cos \chi \mathbf{i} + \sin \chi \mathbf{j}, \quad \mathbf{e}_n = \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{n}} = \frac{\partial \tilde{x}}{\partial \tilde{n}} \mathbf{i} + \frac{\partial \tilde{y}}{\partial \tilde{n}} \mathbf{j} = -\sin \chi \mathbf{i} + \cos \chi \mathbf{j}, \quad (1)$$

where $\tilde{\mathbf{r}} = \tilde{x}\mathbf{i} + \tilde{y}\mathbf{j}$ is the dimensional position vector, \tilde{x} and \tilde{y} dimensional Cartesian coordinates, \mathbf{i} and \mathbf{j} are the unit vectors in \tilde{x} and \tilde{y} directions, \tilde{s} is the dimensional arc-length measured along the periphery of contour of the rigid cylindrical body \tilde{C} , \tilde{n} is a coordinate normal to \tilde{C} , χ is the angle included between the tangent of curve \tilde{C} and x axis, see Fig. 1. The dimensional surface force $\tilde{\mathbf{F}}$ acting on the cylindrical body due to pressure and shear stress exerted by the surrounding moving fluid on the body can be written as follows:

$$\tilde{\mathbf{F}} = - \int_{\tilde{C}} \tilde{p}(\tilde{s}) \mathbf{e}_n \, d\tilde{s} + \int_{\tilde{C}} \tilde{\tau}(\tilde{s}) \mathbf{e}_s \, d\tilde{s} = \tilde{D} \mathbf{i} + \tilde{L} \mathbf{j}, \quad (2)$$

where \tilde{p} is the dimensional pressure, $\tilde{\tau}$ is the dimensional shear stress tangent to contour \tilde{C} , and \tilde{D} and \tilde{L} are the dimensional drag and lift per unit span, respectively. It was assumed here that the main flow comes from the negative x direction. Taking into account the definitions of unit vectors \mathbf{e}_s and \mathbf{e}_n and carrying out integration by parts in Eq. (2),

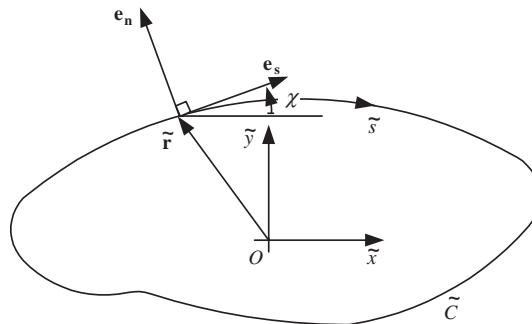


Fig. 1. Main notations for an arbitrary cross-section of a cylindrical body.

force $\tilde{\mathbf{F}}$ can be resolved into drag and lift:

$$\tilde{D} = - \int_{\tilde{C}} \frac{\partial \tilde{p}}{\partial \tilde{s}} \tilde{y}(\tilde{s}) \, d\tilde{s} + \int_{\tilde{C}} \tilde{\tau}(\tilde{s}) \frac{\partial \tilde{x}}{\partial \tilde{s}} \, d\tilde{s}, \quad (3)$$

$$\tilde{L} = \int_{\tilde{C}} \frac{\partial \tilde{p}}{\partial \tilde{s}} \tilde{x}(\tilde{s}) \, d\tilde{s} + \int_{\tilde{C}} \tilde{\tau}(\tilde{s}) \frac{\partial \tilde{y}}{\partial \tilde{s}} \, d\tilde{s}. \quad (4)$$

Please note that all quantities in Eqs. (3) and (4) are dimensional. By introducing scales like l , velocity U and density ρ , all quantities can be nondimensionalized. The dimensionless time t , coordinates x or y , s , pressure p , and shear stress τ can be written as

$$t = \frac{\tilde{t}U}{l}, \quad x = \frac{\tilde{x}}{l}, \quad y = \frac{\tilde{y}}{l}, \quad s = \frac{\tilde{s}}{l}, \quad p = \frac{\tilde{p}}{\rho U^2}, \quad \tau = \frac{\tilde{\tau}}{\rho U^2}, \quad (5)$$

where \tilde{t} is the dimensional time.

By dividing Eqs. (3) and (4) by $\rho U^2/2$ and taking into account definitions (5) of the dimensionless quantities, the drag C_D^* and lift C_L^* coefficients, defined in an inertial system fixed to the stationary cylindrical body, are obtained as follows:

$$C_D^* = \frac{2\tilde{D}}{\rho U^2 l} = -2 \int_C \frac{\partial p}{\partial s} y(s) \, ds + 2 \int_C \tau(s) \frac{\partial x}{\partial s} \, ds, \quad (6)$$

$$C_L^* = \frac{2\tilde{L}}{\rho U^2 l} = 2 \int_C \frac{\partial p}{\partial s} x(s) \, ds + 2 \int_C \tau(s) \frac{\partial y}{\partial s} \, ds, \quad (7)$$

where C is the modified boundary curve of the cylindrical body. C is obtained by dividing the linear dimensions of curve \tilde{C} by length scale l .

Let us investigate now the contribution of pressure p to the lift and drag coefficient in detail in an inertial system. Let us write the dimensionless form of the Navier-Stokes equations in this system for incompressible constant-property Newtonian fluid

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{v}, \quad (8)$$

where \mathbf{v} is the velocity vector nondimensionalized by velocity scale U , t dimensionless time is defined by equation (5), ∇ is the dimensionless del operator, ∇^2 is the Laplace operator, and Re is the Reynolds number defined by $\text{Re} = Ul/\nu$, where ν is the kinematic viscosity of the fluid. In equation (8) the body force is included in the pressure term. Assuming no-slip boundary condition on the cylinder surface and multiplying Eq. (8) by unit vector \mathbf{e}_s we obtain

$$\frac{\partial p}{\partial s} = \frac{1}{\text{Re}} \nabla^2 \mathbf{v} \cdot \mathbf{e}_s. \quad (9)$$

Before deriving the contributions to the drag and lift coefficients due to pressure p using Eqs. (6), (7) and (9), let us investigate the right-hand side (r.h.s.) of Eq. (9) in detail:

$$\nabla^2 \mathbf{v} \cdot \mathbf{e}_s = \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) \frac{\partial x}{\partial s} + \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) \frac{\partial y}{\partial s}.$$

Rearranging this equation yields

$$\begin{aligned} \nabla^2 \mathbf{v} \cdot \mathbf{e}_s &= \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \frac{\partial x}{\partial s} + \left(\frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_x}{\partial x^2} \right) \frac{\partial x}{\partial s} \\ &\quad + \frac{\partial}{\partial x} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \frac{\partial y}{\partial s} + \left(\frac{\partial^2 v_x}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial y^2} \right) \frac{\partial y}{\partial s}. \end{aligned} \quad (10)$$

Taking into account that in the case of two-dimensional flow the vorticity ζ can be written

$$\zeta = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad (11)$$

and that the second and fourth terms on the r.h.s. of Eq. (10) are the x and y derivatives of dilation Θ

$$\Theta = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y},$$

respectively, which vanishes for incompressible medium, further bearing in mind relations (1), we finally obtain

$$\nabla^2 \mathbf{v} \cdot \mathbf{e}_s = -\frac{\partial \zeta}{\partial n}. \quad (12)$$

Hence, comparing Eqs. (9) and (12) yields the pressure derivative along the cylinder surface in the following form:

$$\frac{\partial p}{\partial s} = -\frac{1}{\text{Re}} \frac{\partial \zeta}{\partial n}. \quad (13)$$

In this way the first terms in the drag and lift coefficients defined by Eqs. (6) and (7) can be reshaped to give

$$C_D^* = \frac{2}{\text{Re}} \int_{C_L} \frac{\partial \zeta}{\partial n} y(s) ds + 2 \int_{C_L} \tau(s) \frac{\partial x}{\partial s} ds, \quad (14)$$

$$C_L^* = -\frac{2}{\text{Re}} \int_{C_L} \frac{\partial \zeta}{\partial n} x(s) ds + 2 \int_{C_L} \tau(s) \frac{\partial y}{\partial s} ds. \quad (15)$$

As can be seen in Eqs. (14) and (15), the pressure distribution around the body is not needed for the determination of the force exerted by the fluid on the body. This is especially useful when using the stream function-vorticity method where the pressure is normally not computed. Naturally Eqs. (14) and (15) are not new, and several versions of them can be found in the literature, especially for circular cylinders [see e.g. Badr and Dennis (1985), Cheng et al. (2001)]. It can be seen from Eqs. (6), (7), and (13)–(15) that the contribution of pressure p to drag coefficient C_D^* and to lift coefficient C_L^* can be determined by using the vorticity field in the vicinity of the cylinder.

3. Derivation of lift and drag coefficients in a noninertial system

Let us now investigate the forces acting on a moving body. A general motion of the body is assumed, i.e., both linear and angular velocities and accelerations exist. The rotation of the body takes place around the axis, which is perpendicular to the x, y or ξ, η plane and goes through the origin of the coordinate system. Let us fix the coordinate system to the accelerating body. Using the well-known relationship between particles in an inertial and a noninertial system [see e.g. Shames (1982)] the dimensionless Navier-Stokes equations in this noninertial system can be written as

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w} \cdot \nabla) \mathbf{w} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{w} - \mathbf{a}_0 - 2\boldsymbol{\omega} \times \mathbf{w} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_r) - \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_r, \quad (16)$$

where \mathbf{w} is the relative velocity vector measured in the system fixed to the cylinder moving at acceleration \mathbf{a}_0 , $\boldsymbol{\omega} = \omega \mathbf{k}$ is the dimensionless angular velocity vector of the body, \mathbf{k} is a unit vector perpendicular to ξ, η plane, and \mathbf{r}_r is the position vector measured in the noninertial system. In this equation \mathbf{w} and \mathbf{r}_r are nondimensionalized by velocity and length scales U and l , respectively. In Eq. (16) the body force is included in the pressure term, and the dimensionless angular velocity ω and cylinder acceleration \mathbf{a}_0 are defined as follows:

$$\omega = \frac{\tilde{\omega} l}{U}, \quad \mathbf{a}_0 = \frac{\tilde{\mathbf{a}}_0 l}{U^2},$$

where $\tilde{\omega}$ and $\tilde{\mathbf{a}}_0$ are the dimensional angular velocity and cylinder acceleration, respectively. Other dimensionless quantities are the same as in Eq. (8).

Assuming no-slip boundary condition on the moving cylinder surface in the noninertial system, it follows from Eq. (16) that the pressure derivative on the wall can be written as

$$\frac{\partial p}{\partial s} = \frac{1}{\text{Re}} \nabla^2 \mathbf{w} \cdot \mathbf{e}_s - \left[\mathbf{a}_0 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_r) + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_r \right] \cdot \mathbf{e}_s. \quad (17)$$

We can introduce relative vorticity ζ_r in the relative or noninertial system fixed to the moving cylinder as

$$\zeta_r = \frac{\partial w_\eta}{\partial \xi} - \frac{\partial w_\xi}{\partial \eta},$$

where w_ξ and w_η are the ξ and η components of the relative velocity \mathbf{w} . Repeating the derivations we did earlier (see Eqs. (10)–(12)) the first term on the r.h.s. of Eq. (17) can be transformed into

$$\frac{1}{\text{Re}} \nabla^2 \mathbf{w} \cdot \mathbf{e}_s = -\frac{1}{\text{Re}} \frac{\partial \zeta_r}{\partial n}, \quad (18)$$

similarly to Eq. (13). Taking into account the (1) definition of unit vectors \mathbf{e}_s and \mathbf{e}_n and Eq. (18), Eq. (17) can be transformed into

$$\frac{\partial p}{\partial s} = -\frac{1}{\text{Re}} \frac{\partial \zeta_r}{\partial n} - a_{0s} + \frac{\omega^2}{2} \frac{\partial}{\partial s} (r_r^2) + \frac{1}{2} \frac{d\omega}{dt} \frac{\partial}{\partial n} (r_r^2), \quad (19)$$

where $r_r = (\xi^2 + \eta^2)^{1/2}$ is the dimensionless distance between the variable point on contour C and the origin of the system, and the derivatives of r_r^2 with respect to s and n are to be taken along boundary C . Please bear in mind that when the line (surface) integrals are later evaluated along a stationary body in the inertial system, the x , y coordinates are the same as coordinates ξ , η along the corresponding body in the noninertial system. Here a_{0s} is the tangential component of the dimensionless cylinder acceleration \mathbf{a}_0 along curve C that can be written as

$$a_{0s} = \mathbf{a}_0 \cdot \mathbf{e}_s = a_{0x}(t) \frac{\partial \xi}{\partial s} + a_{0y}(t) \frac{\partial \eta}{\partial s}, \quad (20)$$

where a_{0x} and a_{0y} are the x and y components of the cylinder acceleration.

The drag and lift coefficients can also be defined based on the forces acting in the noninertial system as

$$C_D = \frac{2\tilde{D}_r}{\rho U^2 l} = -2 \int_C \frac{\partial p}{\partial s} \eta(s) ds + 2 \int_C \tau_r(s) \frac{\partial \xi}{\partial s} ds, \quad (21)$$

$$C_L = \frac{2\tilde{L}_r}{\rho U^2 l} = 2 \int_C \frac{\partial p}{\partial s} \xi(s) ds + 2 \int_C \tau_r(s) \frac{\partial \eta}{\partial s} ds, \quad (22)$$

where τ_r is the dimensionless shear stress and \tilde{D}_r and \tilde{L}_r are the dimensional drag and lift measured in the noninertial system fixed to the accelerating cylindrical body. As can be seen, lift and drag coefficients defined in inertial and noninertial systems are very similar in form (see Eqs. (6), (7) and (21), (22)).

Considering the pressure derivative along the body surface in the noninertial system defined by Eq. (19), Eqs. (21) and (22) can be reshaped as

$$C_D = \frac{2}{\text{Re}} \int_C \frac{\partial \zeta_r}{\partial n} \eta ds + \int_C \left[-2a_{0s} + \omega^2 \frac{\partial}{\partial s} (r_r^2) + \frac{d\omega}{dt} \frac{\partial}{\partial n} (r_r^2) \right] \eta ds + 2 \int_C \tau_r \frac{\partial \xi}{\partial s} ds, \quad (23)$$

$$C_L = -\frac{2}{\text{Re}} \int_C \frac{\partial \zeta_r}{\partial n} \xi ds + \int_C \left[-2a_{0s} + \omega^2 \frac{\partial}{\partial s} (r_r^2) + \frac{d\omega}{dt} \frac{\partial}{\partial n} (r_r^2) \right] \xi ds + 2 \int_C \tau_r \frac{\partial \eta}{\partial s} ds. \quad (24)$$

Eqs. (23) and (24) give the drag and lift coefficients for a cylindrical body of arbitrary cross-section that can rotate with varying angular velocity ω and move with an arbitrary acceleration \mathbf{a}_0 in an arbitrary flow of an incompressible Newtonian fluid of constant properties. The coordinate system is fixed to the moving cylinder, and all quantities are dimensionless in these equations. It can be seen by comparing Eqs. (14) with (23) and (15) with (24) that in the noninertial system extra terms appear in the expressions of dimensionless drag and lift acting on the body. Naturally these terms originate from inertial forces.

Let us now consider the relationships for the force coefficients in the inertial and the noninertial system fixed to the accelerated cylindrical body.

4. Relationship between drag and lift coefficients in inertial and non-inertial systems

Let us consider two cases.

- (a) *Inertial system fixed to stationary cylindrical body.* In this case Eqs (14) and (15) give the C_D^* drag and C_L^* lift coefficients based on the integral of the shear stress τ and the normal derivative of the vorticity field ζ . Both of these quantities can be determined if the velocity field \mathbf{v} in the vicinity of the cylinder and the properties of the fluid are known. Naturally the surrounding fluid is moving in a way that the flow should be kinematically identical to that of the case when the body is moving (see case (b)).

(b) *Non-inertial system fixed to the accelerating and/or rotating body.* In this case Eqs. (23) and (24) give coefficients C_D and C_L . These expressions also contain the integral of the shear stress τ_r and the normal derivative of the vorticity field ζ_r which are defined in the noninertial system. Again, these quantities can be determined if the relative velocity field \mathbf{w} in the vicinity of the cylinder and the properties of the fluid are known.

Let flows (a) and (b) be kinematically identical. This means that the absolute velocity vector \mathbf{v} is identical to the relative velocity vector \mathbf{w} measured in the noninertial system fixed to the accelerating cylinder. Hence,

$$\mathbf{v}_A = \mathbf{w}_B. \quad (25)$$

It follows from Eq. (25) that vorticities ζ_A and ζ_B , and shear stresses τ_A and τ_{rB} are also identical on the wall of the cylindrical body for the two cases, i.e.,

$$\zeta_A = \zeta_{rB}, \quad \tau_A = \tau_{rB}. \quad (26)$$

Bearing Eqs. (25) and (26) in mind and comparing the pairs of Eqs. (14), (15) and (23), (24), the following relations can be obtained for the force coefficients defined in noninertial and inertial systems:

$$C_D = C_D^* - \int_C \left[-2a_{0s} + \omega^2 \frac{\partial}{\partial s} (r_r^2) + \frac{d\omega}{dt} \frac{\partial}{\partial n} (r_r^2) \right] \eta \, ds, \quad (27)$$

$$C_L = C_L^* + \int_C \left[-2a_{0s} + \omega^2 \frac{\partial}{\partial s} (r_r^2) + \frac{d\omega}{dt} \frac{\partial}{\partial n} (r_r^2) \right] \xi \, ds. \quad (28)$$

We would like to emphasize again that these two equations give the relationships between force coefficients defined in an inertial and in a noninertial system for two flow cases when the flows in the system fixed to the stationary or moving cylindrical body are kinematically identical. Coefficients C_D and C_L defined in the noninertial system contain some extra terms originating from inertial forces compared to coefficients defined in the inertial system C_D^* and C_L^* . The line (or rather surface) integral of the acceleration (see Eqs. (27) and (28)) is often attributed as an added mass term [see e.g. Blevins (1990)].

So far we have said nothing about the cross-sectional shape of the cylindrical body characterized by contour C . It can be arbitrary except for the condition that curve C has to have finite length, in order for the line integral to be evaluated, as can be seen from Eqs. (27) and (28). Let us determine now the integrals of the dimensionless inertial forces appearing in Eqs. (27) and (28) for circular and rectangular cylinders, which are two of the most important cases from a practical point of view.

5. Drag and lift coefficients for a circular cylinder

General equations were derived in the previous section for the relationships between drag and lift coefficients defined in inertial and noninertial systems for otherwise kinematically identical flows. In this section, these general formulae (27) and (28) are applied for a particular case when the two-dimensional body is a circular cylinder (see Fig. 2). Note that everything is dimensionless in the previously mentioned equations and in Fig. 2 as well. The length scale l appearing in Eqs. (5)–(7), which was also used for making terms dimensionless in Eqs. (27) and (28), is chosen to be identical with the cylinder diameter d . This diameter can be chosen to be unity without the loss of generality. The dimensionless x and y coordinates, and elementary arc-length ds along the circle can be written as follows

$$\xi = 0.5 \cos \varphi, \quad \eta = -0.5 \sin \varphi, \quad ds = 0.5 \, d\varphi, \quad (29)$$

where φ is the polar angle shown in Fig. 2. Let us investigate now the derivative of the r_r^2 terms in Eqs. (27) and (28) along the circle with respect to arc-length s and normal n :

$$\left[\frac{\partial}{\partial s} (r_r^2) \right]_{r_r=0.5} = 0, \quad \left[\frac{\partial}{\partial n} (r_r^2) \right]_{r_r=0.5} = \left[\frac{\partial}{\partial r} (r_r^2) \right]_{r_r=0.5} = 1.0. \quad (30)$$

The s component of cylinder acceleration a_{0s} based on Eqs. (20) and (29) can be written as

$$a_{0s} = -a_{0x}(t) \sin \varphi - a_{0y}(t) \cos \varphi. \quad (31)$$

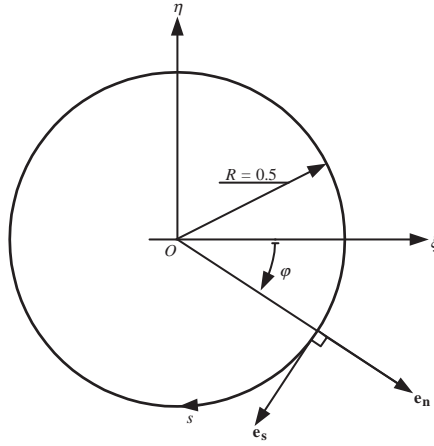


Fig. 2. Notations for a circular cylinder.

Taking into account Eqs. (29)–(31) and using some elementary algebra, Eqs. (27) and (28) will be transformed into the following simple relations

$$C_D = C_D^* + \frac{\pi}{2} a_{0x}, \quad C_L = C_L^* + \frac{\pi}{2} a_{0y}. \quad (32)$$

It can be seen from these equations that the only difference between the drag and lift coefficients considered in inertial and noninertial systems is in the linear acceleration of the body. Neither the angular velocity nor the angular acceleration plays any role in the relationships, due to symmetry of the circular cylinder. In Bearman et al. (1995) a relationship is written for the dimensional lift forces in an inertial system and in a noninertial system oscillating transversely with the circular cylinder. Nondimensionalizing that equation results in the second relationship in Eq. (32).

6. Drag and lift coefficients for a rectangular cylinder

Another important cross-sectional shape of a cylinder is the rectangle. Let us investigate now the relationship between drag and lift coefficients in an inertial and a noninertial system for kinematically identical flows for a rectangular cylinder (see Fig. 3). The origin O of the system around which the body is rotated is chosen to be general. The dimensionless coordinates of corner point A in Fig. 3 will be denoted by ξ_A and η_A . Note that lengths are made dimensionless in Fig. 3 by length scale $l = H$, where H is the height of the cross-section of the rectangular cylinder. The same length scale was also used to make terms dimensionless in Eqs. (27) and (28) based on definitions given in Eq. (5). This height H can be chosen to be unity without loss of generality. Line integrals in Eqs. (27) and (28) are evaluated in clockwise direction as shown in the figure. Arc-length s , normal n , and ξ , η coordinates can easily be identified based on the figure, e.g. for the upper side of the rectangle it is true that

$$ds = d\xi, \quad dn = d\eta, \quad [\eta]_u = 1 + \eta_A, \quad \left[\frac{\partial}{\partial \xi} (r_r^2) \right]_u = 2\xi, \quad \left[\frac{\partial}{\partial n} (r_r^2) \right]_u = 2(1 + \eta_A),$$

where subscript u stands for the upper side of the rectangle shown in Fig. 3. Substituting these expressions in Eqs. (27) and (28), the integrals for the upper line of the rectangle can be evaluated analytically. By localizing variables s , n , ξ and η for the other sides of the rectangle, integrals in Eqs. (27) and (28) can be evaluated for all sides of the rectangle to give

$$C_D = C_D^* + 2a_{0x} \frac{D}{H} - \omega^2 \left(2\xi_A + \frac{D}{H} \right) \frac{D}{H} - 3 \frac{d\omega}{dt} (1 + 2\eta_A) \frac{D}{H}, \quad (33)$$

$$C_L = C_L^* + 2a_{0y} \frac{D}{H} - \omega^2 (1 + 2\eta_A) \frac{D}{H} + 3 \frac{d\omega}{dt} \left(2\xi_A + \frac{D}{H} \right) \frac{D}{H}. \quad (34)$$

So, Eqs. (33) and (34) give the relations for the lift and drag coefficients defined in noninertial and inertial systems where the flows measured from the systems fixed to the stationary or moving cylindrical body are kinematically identical. In these equations the relative position of the rectangle and the point of rotation is characterized by coordinates ξ_A and η_A

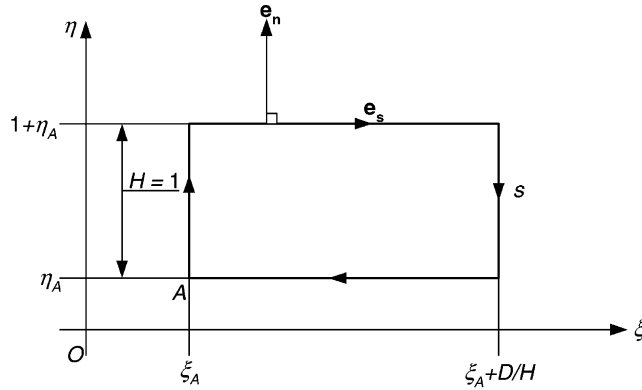


Fig. 3. Notations for a rectangular cylinder.

of point A (see Fig. 3). Naturally the relationships depend on the selection of the point of rotation. It can be seen from these equations that the difference between the drag and lift coefficients considered in inertial and noninertial systems is due to linear and angular accelerations and angular velocity of the body. By selecting these coordinates relationships for special cases can be obtained.

For example if the point of origin (and center of rotation) O coincides with the centroid of area of the rectangle, the coordinates of corner point A shown in Fig. 3 can be written as $\xi_A = -D/(2H)$, $\eta_A = -0.5$. In this case (33) and (34) reduce to the following equations:

$$C_D = C_D^* + 2a_{0x} \frac{D}{H}, \quad (35)$$

$$C_L = C_L^* + 2a_{0y} \frac{D}{H}. \quad (36)$$

So, when a rectangular cylinder is rotated around an axis perpendicular to ξ, η plane and goes through the centroid of area of the cross-section, relations (35) and (36) will be similar to those for a circular cylinder (32). Again, the only difference between the drag and lift coefficients considered in inertial and noninertial systems is due to linear acceleration of the body. Neither the angular velocity nor the angular acceleration plays any role in the relationships, due to symmetry of the rectangular cylinder. These formulae are valid for rectangular cylinders of arbitrary slenderness ratio D/H .

Another special case of practical importance can be when the point of rotation O coincides with corner point A , i.e. $\xi_A = \eta_A = 0$. Substituting these values into Eqs. (33) and (34) the following relationships will be obtained:

$$C_D = C_D^* + 2a_{0x} \frac{D}{H} - \omega^2 \left(\frac{D}{H} \right)^2 - 3 \frac{d\omega}{dt} \frac{D}{H}, \quad (37)$$

$$C_L = C_L^* + 2a_{0y} \frac{D}{H} - \omega^2 \frac{D}{H} + 3 \frac{d\omega}{dt} \left(\frac{D}{H} \right)^2. \quad (38)$$

For a nonrotating case, Eqs. (37) and (38) reduce to Eqs. (35) and (36). If the cross-section of the cylinder is a square, i.e. $D = H$, then Eqs. (33)–(38) simplify further.

7. Conclusions

In this paper, the relationship has been derived for lift and drag coefficients in two cases, (a) and (b).

- an inertial system fixed to a stationary cylindrical body. The motion of the fluid in which the body is placed is an arbitrary function of time not identically zero with a velocity either uniform in space or a linear function of distance from the origin, e.g. can have linear and angular acceleration, such as oscillation or rotation.
- A noninertial system fixed to an accelerating cylindrical body. The relative flow between fluid and body is the same as in (a).

Although the relative motion between the body and the fluid in (a) and (b) for which the formulae are derived is identical kinematically, the forces acting on the bodies differ in the two systems. This is due to inertial forces that occur in a noninertial system. The formulae derived give the relationships between the two systems for each set of coefficients, i.e. the relationship between the lift coefficients for (a) and (b), and the same for the drag coefficient.

The fluid is assumed to be an incompressible constant-property Newtonian fluid; the relative motion between the body and the fluid is assumed to be two-dimensional and to take place in a plane perpendicular to the axis of the body. Three-dimensional effects are ignored, thus limiting the validity of the formulae to low Reynolds number flows. An arbitrary linear and angular acceleration of the cylinder in (b) is assumed.

Two general formulae were derived for the drag and lift coefficients in the two systems for a cylindrical body of arbitrary cross-section. As an example, the formula is applied to a circular cylinder and to a rectangular cylinder, as the two most studied shapes, and the relationship was derived for these two specific cases. Interestingly, we found that when these bodies are rotated about their centroids of area there is no force change between inertial and noninertial systems due to rotation. This is probably true for any body with a symmetrical cross-section about orthogonal axes, although it has not been proved mathematically here.

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